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LETTER TO THE EDITOR

A note on the linearisation of supergravity

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Abstract. The linearised theories of supergravity and superconformal gravity are obtained by directly linearising a second-order superspace. The method gives a clear distinction between the two theories and shows how the unification of the local spacetime symmetries occurs.

The approach to the linearised versions of supergravity and superconformal gravity given by Ferrara and Zumino (1978) has several features in common with the standard superspace approach to supersymmetric Yang–Mills theories (for a review see e.g. Fayet and Ferrara 1977, Wess 1977)‡. In particular the basic potential is a vector multiplet (although with an additional Lorentz vector index), and the generalised gauge transformations may be used to set the physically redundant component fields to zero while giving the linearised versions of superconformal transformations on the gauge fields. (In the pure supergravity case there are also some gauge invariant auxiliary fields which have trivial equations of motion (Stelle and West 1978, Ferrara and van Nieuwenhuizen 1978).) Unlike the Yang–Mills case, where there is only one true gauge invariance, in conformal supergravity there are a number of local x space symmetries, i.e. translations, Lorentz transformations, scale invariance, chiral invariance, and Q and S supersymmetries (Ferrara *et al* 1978, Kaku *et al* 1977). It is a particularly interesting aspect of the approach of Ferrara and Zumino (1978) that all these invariances are contained within one gauge parameter multiplet, although, as one might expect, the gauge transformation is slightly more complicated than for the Yang–Mills case. In this letter we show that the theory may be obtained directly by linearising a curved 'second-order' superspace, and that the unification of parameters occurs in a natural way. We are also able to make a clear distinction between the conformal case and ordinary supergravity. Our method is similar in spirit to that of Wess (1977) and Grimm *et al* (1978a) where the standard superspace formalism for Yang–Mills is obtained from a differential form starting point by imposing constraints on the super Yang–Mills field strength tensor. As we shall see, however, it is the translational gauge field strength which is constrained in supergravity, and its expression in terms of the basic gauge field (linearised *supervierbein*) is somewhat different from the ordinary gauge case.

We begin with a Wess–Zumino curved superspace (Wess 1977, Wess and Zumino 1977, 1978, Grimm *et al* 1978b) with an 'orthonormal' basis of 1-forms $\hat{E}^A$ which differ

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‡ The vector superfield approach to supergravity has also been discussed by Ogietevsky and Sokatchev (1978) and, more recently, by Siegel and James Gates Jr (1978).

only slightly from flat superspace:

$$\hat{E}^A = dz^M \hat{E}_M{}^A, \qquad \hat{E}_M{}^A = E_M{}^A + H_M{}^A. \tag{1}$$

Here $E_M{}^A$ is the standard flat *supervierbein* and $H_M{}^A$ the linearised field. Since it is always possible to choose the flat connection to be zero, we may write for the linearised connection

$$\Phi_{BC}{}^A = E_B{}^M \hat{\Omega}_{MC}{}^A = E_B{}^M (\hat{\Omega}_{MC}{}^A - \Omega_{MC}{}^A) \tag{2}$$

where $\hat{\Omega}_{MB}{}^A$ is the connection of the full space. Next, we introduce the gauge field strengths: defining $K_{BC}{}^A = \hat{T}_{BC}{}^A - T_{BC}{}^A$, $H_B{}^A = E_B{}^M H_M{}^A$, one has

$$\begin{aligned} K_{BC}{}^A = {} & D_B H_C{}^A - (-1)^{bc} D_C H_B{}^A + T_{BC}{}^D H_D{}^A \\ & - H_B{}^D T_{DC}{}^A + (-1)^{bc} H_C{}^D T_{DB}{}^A + \Phi_{BC}{}^A - (-1)^{bc} \Phi_{CB}{}^A \end{aligned} \tag{3}$$

while the linearised curvature (the flat curvature is zero) is

$$R_{ABC}{}^D = D_A \Phi_{BC}{}^D - (-1)^{ab} D_B \Phi_{AC}{}^D + T_{AB}{}^F \Phi_{FC}{}^D. \tag{4}$$

These fields satisfy the linearised Bianchi identities

$$R_{ABC}{}^D - D_A K_{BC}{}^D - T_{AB}{}^F K_{FC}{}^D - K_{AB}{}^F T_{FC}{}^D + \text{cyc.} = 0$$

$$D_A R_{BCD}{}^F + T_{AB}{}^G R_{GCD}{}^F + \text{cyc.} = 0$$

where '+cyc.' means with the cyclically permuted terms added. The field strengths (3) and (4) are invariant under the linearised versions of super-coordinate and local Lorentz transformations, i.e.

$$H_B{}^A \to H_B{}^A + D_B \xi^A + \xi^C T_{CB}{}^A, \qquad \Phi_{BC}{}^A \to \Phi_{BC}{}^A \tag{5}$$

$$H_B{}^A \to H_B{}^A + L_B{}^A, \qquad \Phi_{BC}{}^A \to \Phi_{BC}{}^A - D_B L_C{}^A. \tag{6}$$

In comparison with the Yang–Mills case, clearly the curvature is formally very similar to the Yang–Mills field strength F_{AB}. However, the constraints applicable there (i.e. $F_{\alpha\beta} = F_{\dot{\alpha}\dot{\beta}} = F_{\alpha\dot{\beta}} = 0$) are not in our case. For example one has

$$R_{\alpha\beta,\gamma\delta} = 4(\epsilon_{\alpha\gamma}\epsilon_{\beta\delta} + \epsilon_{\alpha\delta}\epsilon_{\beta\gamma})R^+ \tag{7}$$

where R^+ is one of the three basic superfields used to describe the geometry of a second-order superspace (Wess 1977, Wess and Zumino 1977, 1978, Grimm *et al* 1978b). Instead one has the standard constraints on the supertorsion, and an independent set in terms of the linearised field $K_{BC}{}^A$ may be taken to be

$$K_{\beta\gamma}{}^a = 0, \qquad K_{\beta\dot{\gamma}}{}^a = K_{\beta\dot{\gamma}}{}^{\dot{\alpha}} = K_{\beta\gamma}{}^{\alpha} = 0, \qquad (\sigma^c)_{\dot{\gamma}}{}^{\beta} K_{\beta c}{}^a = 0, \qquad K_{bc}{}^a = 0. \tag{8}$$

The constraints (8) are also invariant under linearised super-Weyl transformations (Howe and Tucker 1978):

$$\delta H_b{}^a = \delta_b^a(\Sigma + \Sigma^+), \qquad \delta H_\beta{}^\alpha = \delta_\beta^\alpha(2\Sigma^+ - \Sigma), \qquad \delta H_b{}^\alpha = -\tfrac{1}{2}\mathrm{i}(\bar{\sigma}_b)^{\dot{\alpha}\alpha}\bar{D}_{\dot{\alpha}}\Sigma^+$$

$$\delta\Phi_{\alpha,\beta\gamma} = -\epsilon_{\alpha\gamma}D_\beta\Sigma - \epsilon_{\alpha\beta}D_\gamma\Sigma, \qquad \delta\Phi_{a,bc} = \eta_{ab}D_c(\Sigma+\Sigma^+) - \eta_{ac}D_b(\Sigma+\Sigma^+) \tag{9}$$

where $\bar{D}_{\dot{\alpha}}\Sigma = 0$.

Our next step is solve (8) for $H_B{}^A$ and $\Phi_{BC}{}^A$. The first equation admits the 'pure gauge' solution

$$H_\beta{}^a = -\mathrm{i}D_\beta V^a - 2\mathrm{i}(\sigma^a)_{\beta\dot{\alpha}}\bar{\phi}^{\dot{\alpha}}, \qquad H_\beta{}^{\dot{\alpha}} = D_\beta\bar{\phi}^{\dot{\alpha}} \tag{10}$$

which is unaltered by transformations of the form

$$V_{\alpha\dot{\beta}} \to V_{\alpha\dot{\beta}} + \bar{S}_{\dot{\beta}\alpha}, \qquad \bar{\phi}_{\dot{\alpha}} \to \bar{\phi}_{\dot{\alpha}} - \tfrac{1}{8}D^{\beta}\bar{S}_{\dot{\alpha}\beta} \tag{11}$$

provided that

$$D_{\alpha}\bar{S}_{\dot{\gamma}\beta} + D_{\beta}\bar{S}_{\dot{\gamma}\alpha} = 0. \tag{12}$$

Hence the transformations on V and ϕ which reproduce (5) for $H_{\beta}{}^{a}$, $H_{\beta}{}^{\dot{\alpha}}$ are

$$V_{\alpha\dot{\beta}} \to V_{\alpha\dot{\beta}} + \bar{S}_{\dot{\beta}\alpha} + \mathrm{i}\xi_{\alpha\dot{\beta}}, \qquad \bar{\phi}_{\dot{\alpha}} \to \bar{\phi}_{\dot{\alpha}} + \bar{\xi}_{\dot{\alpha}} - \tfrac{1}{8}D^{\beta}\bar{S}_{\dot{\alpha}\beta}, \qquad \xi_{\alpha\dot{\beta}} = (\sigma^{a})_{\alpha\dot{\beta}}\xi_{a}. \tag{13}$$

By means of a ξ_{α} transformation we can therefore go to a gauge where $\phi^{\alpha} = 0$ (unlike the Yang–Mills case we may assume hermiticity throughout), while we may use $\xi_{\alpha\dot{\beta}}$ to obtain $V_{\alpha\dot{\beta}}$ hermitian. We then have

$$H_{\beta}{}^{a} = -\mathrm{i}D_{\beta}V^{a}, \qquad V^{a} = (V^{a})^{+}, \qquad H_{\beta}{}^{\dot{\alpha}} = 0 \tag{14}$$

and the amended transformation

$$V_{\alpha\dot{\beta}} \to V_{\alpha\dot{\beta}} + \tfrac{1}{2}(S_{\alpha\dot{\beta}} + \bar{S}_{\dot{\beta}\alpha}) \tag{15}$$

which reproduces (5) for the special parameters

$$\xi_{\alpha\dot{\beta}} = \tfrac{1}{2}\mathrm{i}(\bar{S}_{\dot{\beta}\alpha} - S_{\alpha\dot{\beta}}), \qquad \xi_{\alpha} = \tfrac{1}{8}\bar{D}^{\dot{\beta}}S_{\alpha\dot{\beta}}. \tag{16}$$

Clearly, in view of (12), we may take

$$S_{\alpha\dot{\beta}} = 2\bar{D}_{\dot{\beta}}\Lambda_{\alpha} \tag{17}$$

so that (15) becomes identical to the basic gauge transformation of Ferrara and Zumino (1978). To make further progress, we may use local Lorentz invariance (6) to obtain

$$H_{\beta}{}^{\alpha} = \delta^{\alpha}_{\beta}A. \tag{18}$$

Then the remaining constraint equations may be used to solve the rest of the components of $H_{B}{}^{A}$ and $\phi_{BC}{}^{A}$ in terms of V^{a} and A. One finds

$$\begin{aligned}
&H_{b}{}^{a} = \delta^{a}_{b}(A + A^{+}) + \tfrac{1}{4}(\bar{\sigma}_{b})^{\dot{\beta}\alpha}[\bar{D}_{\alpha}, \bar{D}_{\dot{\beta}}]V^{a}\\
&(\sigma^{b})_{\alpha\dot{\beta}}H_{b}{}^{\dot{\alpha}} = \mathrm{i}\delta^{\dot{\alpha}}_{\dot{\beta}}D_{\alpha}A + \tfrac{1}{8}\mathrm{i}DD\bar{D}_{\dot{\beta}}V_{\alpha}{}^{\dot{\alpha}}\\
&\Phi_{\alpha,\beta\gamma} = \epsilon_{\alpha\gamma}D_{\beta}A + \epsilon_{\alpha\beta}D_{\gamma}A\\
&\Phi_{a,bc} = \tfrac{1}{2}D_{a}(H_{bc} - H_{cb}) - \tfrac{1}{2}D_{b}(H_{ac} + H_{ca}) + \tfrac{1}{2}D_{c}(H_{ab} + H_{ba})\\
&\Phi_{\alpha,\dot{\beta}}{}^{\dot{\gamma}} = \delta^{\dot{\gamma}}_{\dot{\beta}}D_{\alpha}(2A + A^{+}) + \tfrac{1}{4}DD\bar{D}_{\dot{\beta}}V_{\alpha}{}^{\dot{\alpha}}.
\end{aligned} \tag{19}$$

Since $\Phi_{\alpha,\dot{\beta}}{}^{\dot{\beta}} = 0$, the last of these equations may be solved for A:

$$A = -\tfrac{1}{6}D_{\alpha}\bar{D}_{\dot{\beta}}V^{\alpha\dot{\beta}} - \tfrac{1}{12}\bar{D}_{\dot{\beta}}D_{\alpha}V^{\alpha\dot{\beta}} \tag{20}$$

up to a chiral superfield which is a reflection of the super-Weyl invariance. Under a gauge transformation (15) one finds

$$A \to A + \tfrac{1}{2}D_{\alpha}\xi^{\alpha} + 2\Sigma^{+} - \Sigma \tag{21}$$

where ξ_{α} is given by (16) and

$$\Sigma = \tfrac{1}{12}\bar{D}\bar{D}D^{\alpha}\Lambda_{\alpha} \tag{22}$$

is clearly chiral. It is then straightforward to check that the basic gauge transformation reproduces the transformations (5), (6) and (9) if the fields are given by (19) and the parameters by (16) and (22), as long as

$$L_{\alpha\beta} = \tfrac{1}{8}D_\alpha\bar{D}\bar{D}\Lambda_\beta + \tfrac{1}{8}D_\beta\bar{D}\bar{D}\Lambda_\alpha. \tag{23}$$

This last equation is just the condition that we remain in the gauge (18). The various linearised tensors of supergauge may then be computed directly from (3) and are given by the same formulae of Ferrara and Zumino (1978) up to a normalisation factor. For example,

$$R^+ = \tfrac{1}{24}\mathrm{i}DD\ \partial_{\alpha\dot{\beta}}V^{\alpha\dot{\beta}}. \tag{24}$$

Hence we have shown that, at the linearised level, superspace is described by the hermitian multiplet $V_{\alpha\dot{\beta}}$ undergoing the gauge transformation (15). Furthermore, the linearised gauge parameters are clearly contained in Λ_α via equations (16), (22) and (23). It is also clear that supergravity transformations are obtained from superconformal transformations by requiring $\Sigma = 0$, where Σ is given by (22). The equations of motion of linearised supergravity may be obtained by directly linearising the full set (Wess and Zumino 1977, 1978, Grimm *et al* 1978b), while in the conformal case, the action of Howe and Tucker (1978) may be linearised to give the action of Fayet and Ferrara (1977) and Wess (1977) via the Gauss–Bonnet theorem.

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